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ABSTRACT

A general discussion of the properties of magnetohydrodynamic flows at low conductivity is given, and then attention is restricted to the class of such flows satisfying the following conditions:

1. The flow is steady, two-dimensional, inviscid, and only slightly perturbed from uniform conditions.
2. The magnetic field vector is also two-dimensional and lies in the plane of the flow.
3. The distortion of the applied field by the induced currents is negligible.
4. Physical boundaries on the flow are one or two infinite plates parallel to the flow direction.
5. The conductivity of the fluid is a scalar quantity, but may vary in a restricted manner with position.

With these assumptions, the perturbations to the flow are calculated exactly for arbitrary magnetic fields for the cases in which the undisturbed flow is either incompressible or supersonic. Illustrative examples for simple magnetic fields are evaluated; for some of these examples the subsonic case is also treated. Another particular example is used to show the effect of spatially varying conductivity on the flow. The limits of the applicability of these results are discussed, and general conclusions regarding the nature of the flow perturbation are drawn.

EXACT SOLUTIONS TO A CLASS OF LINEARIZED MAGNETOHYDRODYNAMIC FLOW PROBLEMS

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I. INTRODUCTION

The study of magnetohydrodynamic flow problems in which the conductivity is small is of interest in such areas as flight magnetohydrodynamics,[1,2] magnetohydrodynamic power generation,[3,4] magnetohydrodynamic pumps and flow meters[5] and some shock tube work.[6] Although much work has been done on these subjects, there exist surprisingly few theoretical solutions applicable to this type of problem; this is in marked contrast to the situation in which the conductivity is taken to be infinite.

Good discussions of the features that distinguish the low conductivity problems are to be found in the works of Kemp and Petschek[7] and Hains, et al.[8] The most important of these features is the small value of the magnetic Reynolds number. The effect of this is that the applied magnetic

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[1]Resler, Jr., E. L. and Sears, W. R., J. Aeronaut. Sci. 25, 235-245 (1958).

[2]Rosa, R. J., Ph.D. Thesis, Cornell University, Ithaca, New York (1956).

[3]Rosa, R. J., Phys. Fluids 4, 182 (1961).

[4]Sutton, George W., ARS Preprint 2005-61 (1961).

[5]Shercliff, J. A., United Kingdom Atomic Energy Research Establishment, Report No. X/R 1052 (1953)

[6]Patrick, R. H. and Brogan, T. R., J. Fluid Mech. 5, 289 (1959).

[7]Kemp, N. H. and Petschek, H. E., J. Fluid Mech. 4, 553 (1958).

[8]Hains, F. D., Yoler, Y. A., and Ehlers, E., Proceedings of the Third Biennial Gas Dynamics Symposium (1959) (Northwestern University Press, Evanston, Illinois, Dynamics of Conductivity Gases).

field is only slightly distorted by the flow in the region of interest. In addition to the magnetic Reynolds number, a further parameter (called the interaction parameter, see Eq. (9)) is needed to describe the perturbation of the flow from some initial (usually uniform) state.

The procedure of Kemp and Petschek is to write the various flow and electromagnetic quantities as the sum of zeroth order quantities and two perturbation terms proportional respectively to the magnetic Reynolds number and the interaction parameter and then to solve the resultant linear equations for the perturbation quantities. This general analysis will be assumed in the present paper, and attention will be confined to the determination of those perturbations of the flow quantities which are proportional to the interaction parameter. Similar treatments of the problem are also given by Ehlers[9] (for the axisymmetric case) and by Morioka[10] (for the two-dimensional case).

The geometrical configurations to be studied were suggested by two papers of Sherman[11,12] in which a steady two-dimensional uniform slightly conducting incompressible flow passes through a channel; a magnetic field, also two dimensional, is applied to the flow and has the general effect of obstructing it. In the particular case studied by Sherman the magnetic field was due to a current flowing in a straight wire lying outside the channel and perpendicular to the flow. In this paper, an exact solution to this problem is given which agrees closely with the numerical results given by Sherman for the inviscid case. In addition, a wide selection of more involved problems are solved analytically. These include cases in which the applied magnetic field is of arbitrary form (for the incompressible and supersonic cases), as well as subsonic cases for simple magnetic field configurations and cases in which the conductivity (which has hitherto been tacitly considered to be constant) is allowed to vary spatially in a

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[9]Ehlers, F. E., ARS J. 31, 334-342 (1961).

[10]Morioka, S., J. Phys. Soc. Japan 16, 2544-2550 (1961).

[11]Sherman, A., Advances in the Astronautical Sciences, (The MacMillan Company, New York, 1961) Vol. 6.

[12]Sherman, A., Phys. Fluids 4, 552-557 (1961).

restricted manner. On the other hand, no boundary layer calculations based on the computed inviscid pressure gradients are included.

The basic mechanism by which the applied magnetic field retards the flow is as follows: an electromotive force is induced in the flow in the direction perpendicular to the plane containing the velocity and the magnetic field vectors. This electromotive force drives a current through the fluid which is assumed to be returned at infinity. This may be imagined more realistically by considering the two-dimensional channel as the limit of a large annular duct. No externally applied electromotive force is considered to be present, although such an arrangement might be interesting experimentally. In the present small perturbation analysis, the electric current is calculated directly from Ohm's law using the unperturbed values of the velocity and magnetic field. It is a small quantity, being proportional to the conductivity. The effect of the current flowing through the fluid in the presence of a magnetic field is to produce a body force on the fluid, whose general action is to retard the flow. This body force, which is also proportional to the conductivity, may then be calculated as a function of position; and, mathematically, the first order flow problem becomes one in which a given (non-conservative) body force is applied to the fluid. This is the specific problem which is solved in this paper for a wide variety of conditions. It should be noted that in order to calculate the electromagnetic force on the fluid as a whole to the first order in the interaction parameter (or, what is the same thing, its reaction at the coil giving rise to the magnetic field) it is not necessary to calculate the perturbed flow field. However, in the presence of a physical boundary, there is an additional force of the first order in the interaction parameter in the form of an induced pressure which cannot be found without first calculating the flow perturbation.

The nature of the relation between the electric current and the electric field (Ohm's law) in an ionized gas has been extensively discussed by Schluter.[13] Kemp and Petschek (loc. cit.) made use of a simplified

[13]Schluter, A., Z. Naturforschg 5a, 72 (1950) and 6a, 73 (1951).

form of Schluter's result applicable to a slightly ionized gas, but in which nevertheless both Hall and ion slip effects were present. The possibility of using this form of Ohm's law with tensor conductivity without greatly complicating the mathematics arose because of a number of special features of the geometry chosen. In the present case, the use of such an Ohm's law would add considerably to the mathematics and has, therefore, been eschewed in the interests of brevity and clarity. In its stead, the simple form of Ohm's law with scalar conductivity is used. The limit of applicability of this law is roughly that each electron should suffer at least one scattering collision per cyclotron period; it is not hard to predict the general nature of the modification to the present work that would be caused by an important Hall effect.

While the assumptions used in this paper are, in the above respect, more restrictive than those of Kemp and Petschek, in another respect they are less so; for a number of cases of linearized compressible flow, both subsonic and supersonic have been solved and are presented here. The basis for the solution of compressible magnetohydrodynamic flow problems by these techniques may be found in Fishman, et al.[14] It will be sufficient to remark here that, apart from the usual limitations of linearized compressible flow theories, an additional complication arises in view of the Joule heating of the flow by the induced currents within it. This is particularly important at high Mach numbers where the relative change in temperature may be high, leading to large changes in the conductivity. As an example of this effect, Lin[15] predicts that at atmospheric pressure and temperatures around 5000°K the conductivity of air increases as about the 10th power of the temperature. At higher temperatures this effect is considerably less marked.

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[14] Fishman, F., Lothrop, J., Patrick, R. and Petschek, H., Avco-Everett Research Laboratory Report No. 39 (1959).

[15] Lamb, L. and Lin, S. C., J. Appl. Phys. 28, 754 (1957).

II. MATHEMATICAL FORMULATION

On the basis of the remarks of the preceding section, the mathematical formulation of the problem is easily accomplished. Using asterisks to denote dimensional quantities, and taking Cartesian coordinates x^* y^* in the plane containing both the flow and the magnetic field vectors, Ohm's law shows that the electric current is in the z-direction and is given by:

$$j_z^* = \sigma^* (u^* B_y^* - v^* B_x^*) \tag{1}$$

The equations of motion are:

$$\rho^* \left(u^* \frac{\partial u^*}{\partial x^*} + v^* \frac{\partial u^*}{\partial y^*} \right) + \frac{\partial p^*}{\partial x^*} + j_z^* B_y^* = 0 \tag{2}$$

$$\rho^* \left(u^* \frac{\partial v^*}{\partial x^*} + v^* \frac{\partial v^*}{\partial y^*} \right) + \frac{\partial p^*}{\partial y^*} - j_z^* B_x^* = 0 \tag{3}$$

The equation of continuity is:

$$\frac{\partial}{\partial x^*} (\rho^* u^*) + \frac{\partial}{\partial y^*} (\rho^* v^*) = 0 \tag{4}$$

The energy equation may be taken to be:

$$u^* \frac{\partial s^*}{\partial x^*} + v^* \frac{\partial s^*}{\partial y^*} = {j_z^*}^2 / \sigma^* \rho^* T^* \tag{5}$$

Finally, a perfect gas with constant γ is assumed, so that

$$p^* = \rho^* R T^* \tag{6}$$

$$(\gamma - 1) ds^* = R \left[dp^*/p^* - \gamma \, d\rho^*/\rho^* \right] \tag{7}$$

The following non-dimensional quantities are defined:

$$(x, y) = (x^*, y^*)/y_0 \; ; \quad j_z = j_z^*/\sigma_0 U B_0 \quad ; b_x, b_y = B_x^*, B_y^*/B_0 \; ;$$

$$\sigma = \sigma^*/\sigma_0 \; ; \quad u', v' = u^*, v^*/U \quad ; \rho' = \rho^*/\rho_0 \; ;$$

$$s' = s^*/R \; ; \quad p' = p^*/\rho_0 U^2 \tag{8}$$

Then, in terms of the interaction parameter defined by

$$S = \sigma_0 B_0^2 y_0 / \rho_0 U \tag{9}$$

Equations (1) to (7) become (after elimination of the temperature)

$$j_z = \sigma (u' b_y - v' b_x) \tag{10}$$

$$\rho'\left(u' \frac{\partial u'}{\partial x} + v' \frac{\partial u'}{\partial y}\right) + \frac{\partial p'}{\partial x} + S\, j_z b_y = 0 \tag{11}$$

$$\rho'\left(u' \frac{\partial v'}{\partial x} + v' \frac{\partial v'}{\partial y}\right) + \frac{\partial p'}{\partial y} - S\, j_z b_x = 0 \tag{12}$$

$$\frac{\partial}{\partial x}(\rho' u') + \frac{\partial}{\partial y}(\rho' v') = 0 \tag{13}$$

$$u' \frac{\partial s'}{\partial x} + v' \frac{\partial s'}{\partial y} = S\, j_z^2 / p' \tag{14}$$

$$(\gamma - 1)\, ds' = \left[dp'/p' - \gamma\, d\rho'/\rho'\right] \tag{15}$$

Next, the assumption of small perturbations must be invoked. The unperturbed flow is taken to be uniform in the x-direction and perturbation quantities are defined as follows:

$$u' = 1 + Su \qquad v' = Sv \qquad \rho' = 1 + S\rho$$

$$s' = Ss \qquad p' = 1/\gamma M^2 + Sp \tag{16}$$

where $M^2 = \rho_o U^2/\gamma p_o$ is the free stream Mach number. The linearized (first order in S) flow problem to be dealt with is then defined by the following equations:

$$\frac{\partial u}{\partial x} + \frac{\partial p}{\partial x} = -\sigma b_y^2 \tag{17}$$

$$\frac{\partial v}{\partial x} + \frac{\partial p}{\partial y} = \sigma b_x b_y \tag{18}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial \rho}{\partial x} = 0 \tag{19}$$

$$\frac{\partial s}{\partial x} = \gamma M^2 \sigma b_y^2 \tag{20}$$

$$(\gamma - 1) ds = \gamma M^2 dp - \gamma d\rho \tag{21}$$

The specification of the problem is completed by noting that the magnetic field vector must be irrotational (this is a result of assuming the magnetic Reynolds number to be negligible) and, of course, solenoidal. Thus:

$$\frac{\partial b_y}{\partial x} - \frac{\partial b_x}{\partial y} = 0 \, , \quad \frac{\partial b_x}{\partial x} + \frac{\partial b_y}{\partial y} = 0 \tag{22}$$

Before proceeding to the solution of any particular problem, certain general simplifications may be made in the above system of equations. First, the entropy is given directly from Eq. (20) as:

$$s = \gamma M^2 \int_{-\infty}^{x} \sigma b_y^2 \, dx \tag{23}$$

Secondly, Eq. (21) may be integrated immediately to give the density in terms of the entropy (now a known quantity) and the pressure:

$$\rho = M^2 p - \frac{(\gamma - 1)}{\gamma} s \tag{24}$$

Next, the x-component of velocity is given in terms of the (unknown) pressure and the (known) applied magnetic field by:

$$u = -p - \int_{-\infty}^{x} \sigma b_y^2 \, dx \tag{25}$$

Lastly, u, ρ, and s may be eliminated from Eqs. (17), (19) and (24) to give:

$$\frac{\partial v}{\partial y} + (M^2 - 1) \frac{\partial p}{\partial x} = [1 + (\gamma - 1) M^2] \sigma b_y^2 \tag{26}$$

which, together with Eq. (18), forms a system of two inhomogeneous partial differential equations for v and p. If these can be solved the remaining quantities may all be found directly from relations (23), (24) and (25). Although these equations are linear in the unknowns, they are non-linear in the applied magnetic field so that solutions for different field configurations may not be superposed.

III. INCOMPRESSIBLE PROBLEMS

For the present, the conductivity will be assumed constant throughout the flow so that σ may be taken to be unity. Then, for the incompressible case, Eqs. (18) and (26) reduce to

$$\frac{\partial v}{\partial x} + \frac{\partial p}{\partial y} = b_x b_y \tag{27}$$

$$\frac{\partial v}{\partial y} - \frac{\partial p}{\partial x} = b_y^2 \tag{28}$$

This system of linear inhomogeneous partial differential equations is best dealt with by splitting the solution into a particular solution (distinguished by the subscript p) which will depend only on the form of the magnetic field, and a complementary solution (distinguished by the subscript c) which will satisfy the homogeneous part of Eqs. (27) and (28) together with suitable boundary conditions. It is a remarkable fact that when the magnetic field satisfies Eq. (22) and vanishes reasonably at infinity a simple particular integral can always be found; it is

$$v_p = \frac{1}{2} b_y \int_{-\infty}^{x} b_x \, dx \tag{29}$$

$$p_p = -\frac{1}{2} b_x \int_{-\infty}^{x} b_x \, dx + \frac{1}{2} \int_{-\infty}^{x} \left[b_x^2 - b_y^2 \right] dx \tag{30}$$

Furthermore, the complementary solution may be obtained as

$$p_c + i v_c = w(x + iy) = w(z)$$

where w (z) is an analytic function determined by the relevant boundary conditions. In this paper, as indicated in the introduction, the flow will always be bounded by one or two infinite planes parallel to the flow. Thus, the boundary conditions will have the form $Iw + v_p = 0$ for one or two values of y and all values of x.

A number of illustrative examples of these solutions have been

worked out and are presented by Levy.[16] In the interests of brevity only two of these will be reproduced here. In both cases the flow is bounded only by the plane y = 0; the magnetic field in the first case is that due to a current flowing in a single wire at the point (0, -1). Thus, the physical distance of the wire from the boundary is y_o; taking B_o to be the field strength at the origin, $\mu_o I / 2\pi y_o$, the field components become:

$$b_x = \frac{y+1}{x^2+(y+1)^2} \qquad b_y = \frac{-x}{x^2+(y+1)^2} \tag{31}$$

Then

$$v_p = \frac{-\frac{1}{2}x\left[\pi/2 + \tan^{-1}\frac{x}{y+1}\right]}{x^2+(y+1)^2} \tag{32}$$

$$p_p = \frac{\frac{1}{2}\left[x-(y+1)(\pi/2 + \tan^{-1}\frac{x}{y+1})\right]}{x^2+(y+1)^2} \tag{33}$$

The boundary condition on w (z) is

$$\vartheta w = \frac{\frac{1}{2}x\left[\pi/2 + \tan^{-1}x\right]}{x^2+1} \qquad \text{on } y=0 \tag{34}$$

and since w (z) is analytic in the upper half-plane,

$$w(z) = \frac{i\pi/4}{z+i} - \frac{\frac{1}{2}z\ln\frac{1-iz}{2}}{z^2+1} \tag{35}$$

From the above the whole perturbed flow may be found at once. It will be sufficient here to quote the result for the pressure on the wall which is

[16] Levy, R. H., Avco-Everett Research Laboratory, AMP 70 (1962).

$$p = \frac{\frac{1}{2}(x - \tan^{-1}x) - \frac{1}{4}\, x \ln \frac{1+x^2}{4}}{x^2 + 1} \tag{36}$$

Levy (loc. cit.) has also calculated this flow when an additional boundary is placed at y = H. The pressure on the lower wall for various values of H are shown in Fig. 1; the results for H = 6.4 agree fairly well with those of Sherman except for a small constant displacement which is probably due to the fact that in his numerical calculations the pressure was assumed to vanish at some finite negative value of x. The total pressure drop along the channel is $\pi \ln (H + 1)/2H$.

The slow vanishing of the field at large distances in the above case results in the divergence of the drag integral. This difficulty is avoided in the next example, where a linear dipole with axis parallel to the flow is placed at the point (0, -1). If B_o is again taken to be the field at the origin, the components are:

$$b_x = \frac{-2x(y+1)}{[x^2+(y+1)^2]^2} \qquad b_y = \frac{x^2-(y+1)^2}{[x^2+(y+1)^2]^2} \tag{37}$$

Then

$$v_p = \frac{\frac{1}{2}(y+1)[x^2-(y+1)^2]}{[x^2+(y+1)^2]^3} \tag{38}$$

$$p_p = \frac{\frac{1}{2}x[(y+1)^2+\frac{1}{3}x^2]}{[x^2+(y+1)^2]^3} \tag{39}$$

and, on y = 0

$$\vartheta_w = \frac{-\frac{1}{2}(x^2-1)}{(x^2+1)^3} \tag{40}$$

Thus

$$w(z) = -\frac{1}{8}(z^2+3iz-4)(z+i)^{-3} \tag{41}$$

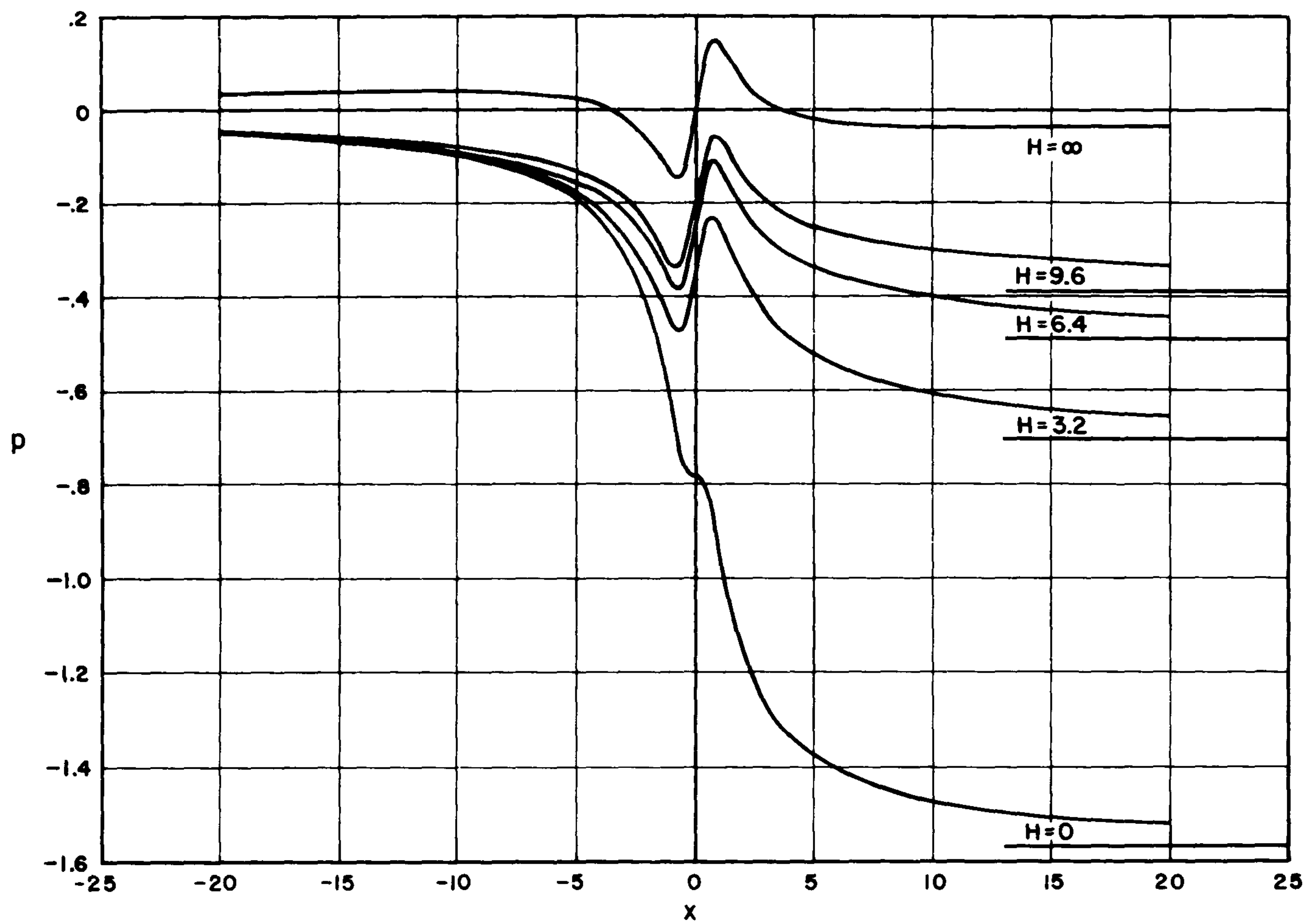

Fig. 1 This is the pressure on the lower wall of a channel of height H when the flow is impeded by the magnetic field due to a current flowing in a single wire at unit distance below the wall.

7194

Hence, the pressure on the wall is

$$p = -\frac{1}{8} x \left[x^4 + \frac{2}{3} x^2 + 5\right]\left[x^2 + 1\right]^{-3} \tag{42}$$

and this is shown, together with the corresponding results for the case of a channel of height H in Fig. 2. The drag per unit length on the dipole for the single wall case is $(\pi/8)\,\rho U^2 y_0 S$.

A very interesting observation may be made in respect to the pressure force exerted on the fixed wall (for the single wall case), the observation being applicable to any magnetic field configuration and not only the two examples given above. The net force, where the field is symmetric about the line $x = 0$ is, of course, zero, but the moment about the origin exerted by the pressure distribution diverges linearly. Mathematically speaking, this is because the pressure returns to zero only as x^{-1} for large x; it is an unexpected result that this behaviour is independent of the form, or the rate of decay of the magnetic field. It may be shown that the result is a consequence of the finite displacement of the streamlines in passing from $x = -\infty$ to $x = +\infty$, and always follows where this displacement effect can be exhibited; the displacement of the streamlines results from the finite velocity defect at $x = +\infty$, which in turn may be deduced from Eq. (25) with the observation that p must return to zero at $x = +\infty$. Of course, in a practical case the moment actually obtained will be limited both by three-dimensional effects and by the finite size of the body; however, it might well turn out that the actual moment might well be more important than the drag for certain flight applications. A last remark in this connection is that the phenomenon does not appear in supersonic flow for the reason that, as will appear, the pressure on the wall at a certain point depends only on the values taken by the magnetic field along the characteristics leading to the point in question. Thus, if the field is confined to a finite region, or decays very rapidly, the pressure will behave similarly.

The method of solving Eqs. (27) and (28) used depends on the existence of the simple form of the particular solution Eqs. (29) and (30). If this particular solution had not been found, Eqs. (27) and (28) would still be amenable to solutions using the more powerful transform techniques.

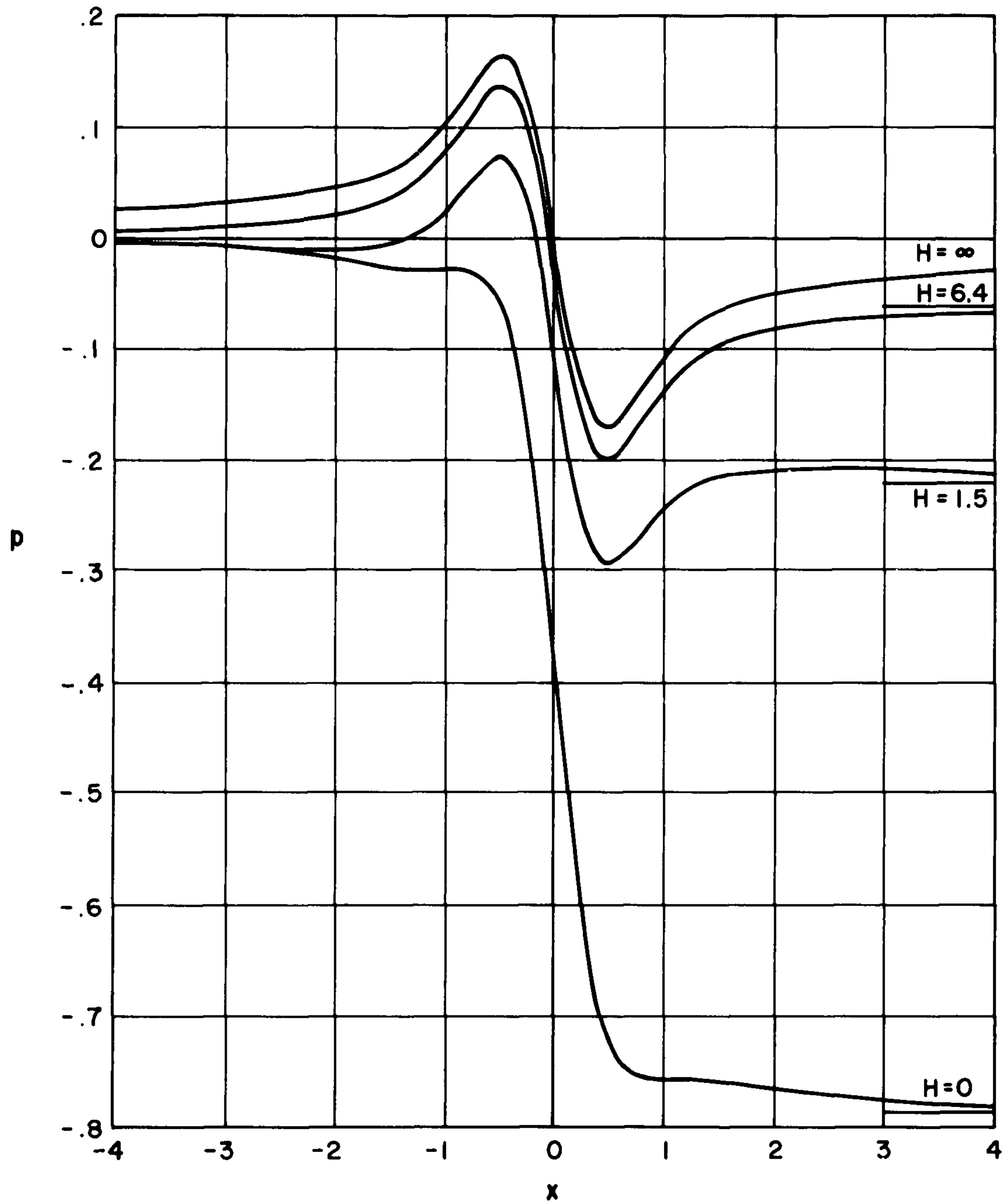

Fig. 2 This is the pressure on the lower wall of a channel of a height H when the flow is impeded by the magnetic field due to a linear dipole at unit distance below the wall.

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However, a great advantage of the present procedure is the simplicity of the results which makes for greater understanding of the problem. It will shortly be seen that the particular solution does not apply to the subsonic cases so that the subsonic results obtained are much more complicated. For the incompressible case, other field configurations are easily studied; and, apart from the results of Levy already referred to, the solution to the interesting case of two wires with finite separation carrying equal and opposite currents is presented in the same paper (loc. cit.).

It appears that simple solutions of the type discussed are also available for axisymmetric flows. From the paper of Ehlers (loc. cit.) the equations corresponding to Eqs. (27) and (28) may be deduced. They are

$$\frac{\partial v}{\partial x} + \frac{\partial p}{\partial r} = b_x b_r \tag{43}$$

$$\frac{1}{r}\frac{\partial}{\partial r}(rv) - \frac{\partial p}{\partial x} = b_r^2 \tag{44}$$

and the magnetic field components now satisfy

$$\frac{\partial b_r}{\partial x} - \frac{\partial b_x}{\partial r} = 0 \, , \quad \frac{\partial b_x}{\partial x} + \frac{1}{r}\frac{\partial}{\partial r}(r b_r) = 0 \tag{45}$$

instead of Eq. (22). The analogous particular solution is then

$$v_p = \frac{1}{2} b_r \int_{-\infty}^{x} b_x \, dx + \int_{-\infty}^{x} b_x b_r \, dx \tag{46}$$

$$p_p = -\frac{1}{2} b_x \int_{-\infty}^{x} b_x \, dx \tag{47}$$

The application of this result to the particular problem discussed by Ehlers in which the flow of a fluid down an infinite pipe is impeded by the magnetic field due to a concentric circular coil is not possible in closed form owing to the complicated form of the field components. However, the numerical solution to this problem would presumably be easier by this method; as above, it involves only the use of v_p calculated on the boundary as the boundary condition in a potential flow problem. It can be shown that for flows along the exterior of an infinite pipe the pressure returns to zero like x^{-2}; in view of the geometry no net moment is exerted by this force.

IV. SUBSONIC PROBLEMS

The equations governing the subsonic case with uniform conductivity are

$$\frac{\partial v}{\partial x} + \frac{\partial p}{\partial y} = b_x b_y \tag{48}$$

$$\frac{\partial v}{\partial y} - (1-M^2)\frac{\partial p}{\partial x} = \left[1+(\gamma-1)M^2\right] b_y^2 \tag{49}$$

If the coordinates and variables are transformed by the Prandtl-Glauert rule the field components will no longer satisfy the relation (22). In these circumstances, no particular solution of these equations has been found for arbitrary magnetic fields. However, solutions may be obtained when the field components are given by Eqs. (31) or (37) for in both these cases transformation to polar coordinates with origin at (0, -1) makes possible the determination of particular solutions by separation of variables. The complementary solutions can again be found by complex variable techniques using the function

$$w_c(z = x + i\beta y) = \beta p_c + i v_c \tag{50}$$

where $\beta = \sqrt{1 - M^2}$ for subsonic flow.

The results for both cases are given in Levy (loc. cit.). It will be sufficient here to quote the result for the pressure on the single wall in the single wire case which is:

$$P(x,0) = \frac{-\frac{1}{2}\left[1+(\gamma-1)M^2\right]}{x^2+\beta^2}\left[\frac{1}{2} \quad \frac{x}{\beta}\ln\frac{1+x^2}{(1+\beta)^2} + \tan^{-1} x\right]$$

$$+\frac{\frac{1}{2}x(\gamma/\beta - \gamma+1)}{x^2+1} \tag{51}$$

This pressure distribution and that for the dipole case are shown in Figs. 3 and 4 for $\gamma = 1.4$. This value has been chosen to emphasize the heating of the gas; however, graphs for $\gamma = 1.0$ (drawn to emphasize the force effect) are given in Levy (loc. cit.). The points to note are first, that in both cases the pressure again dies like x^{-1}, and second, that as in all cases of linearized compressible flow, the pressure change increases markedly as the flow approaches sonic velocity.

Morioka (loc. cit.) has also obtained explicit solutions to the dipole case both for the subsonic flow discussed here and for the supersonic flow to be treated in the next section. In both cases, the results differ in detail from those of Levy. However, it is suspected that the differences may be due to nothing more than misprints.

V. SUPERSONIC PROBLEMS

When the flow is supersonic, the problems under consideration can be treated by the method of characteristics. Introducing

$$\xi = \frac{M}{2\beta}x - \quad \frac{M}{2}(y+1) \; ; \quad \eta = \frac{M}{2\beta}x + \quad \frac{M}{2}(y+1) \tag{52}$$

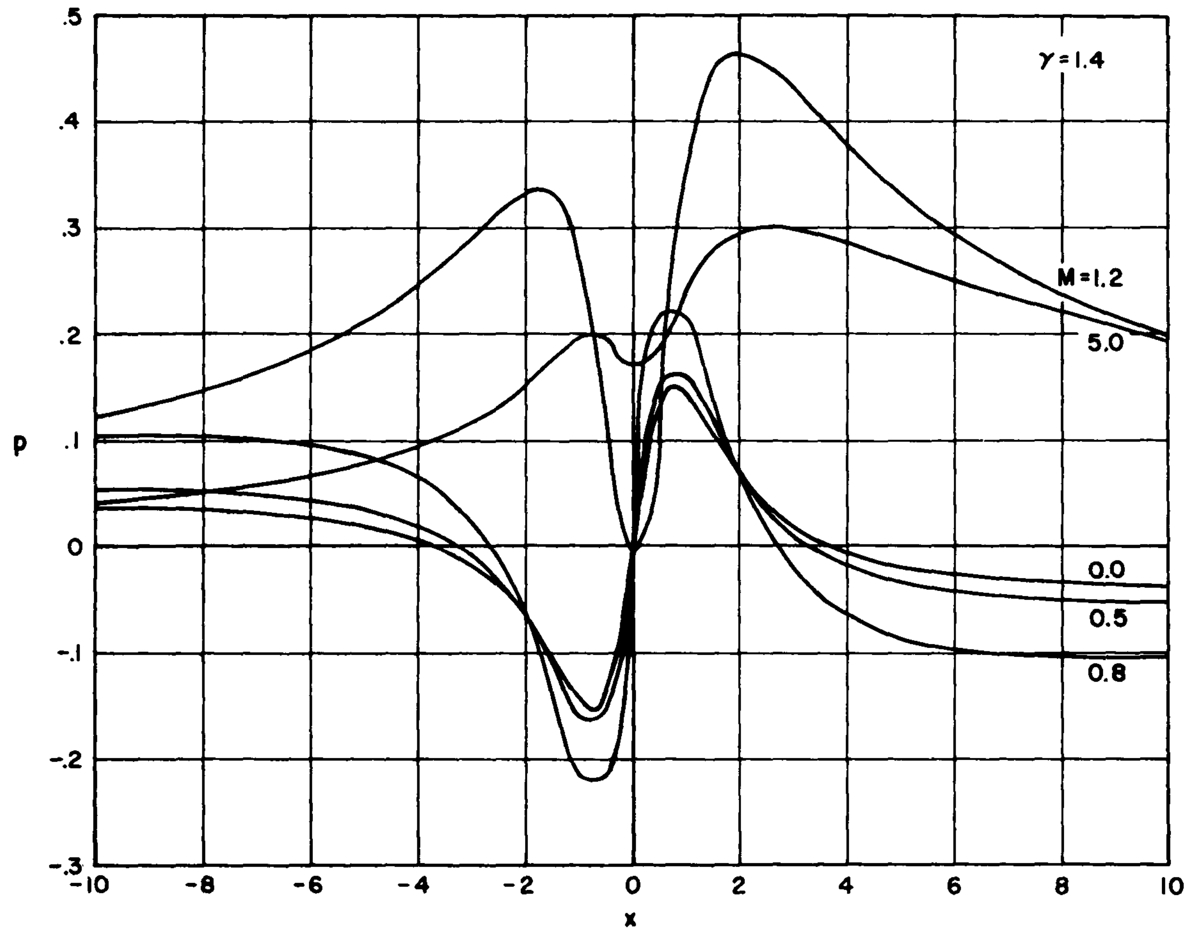

Fig. 3 This is the pressure on a wall when the flow is impeded by the magnetic field due to a current flowing in a single wire at unit distance below the wall. $\gamma = 1.4$.

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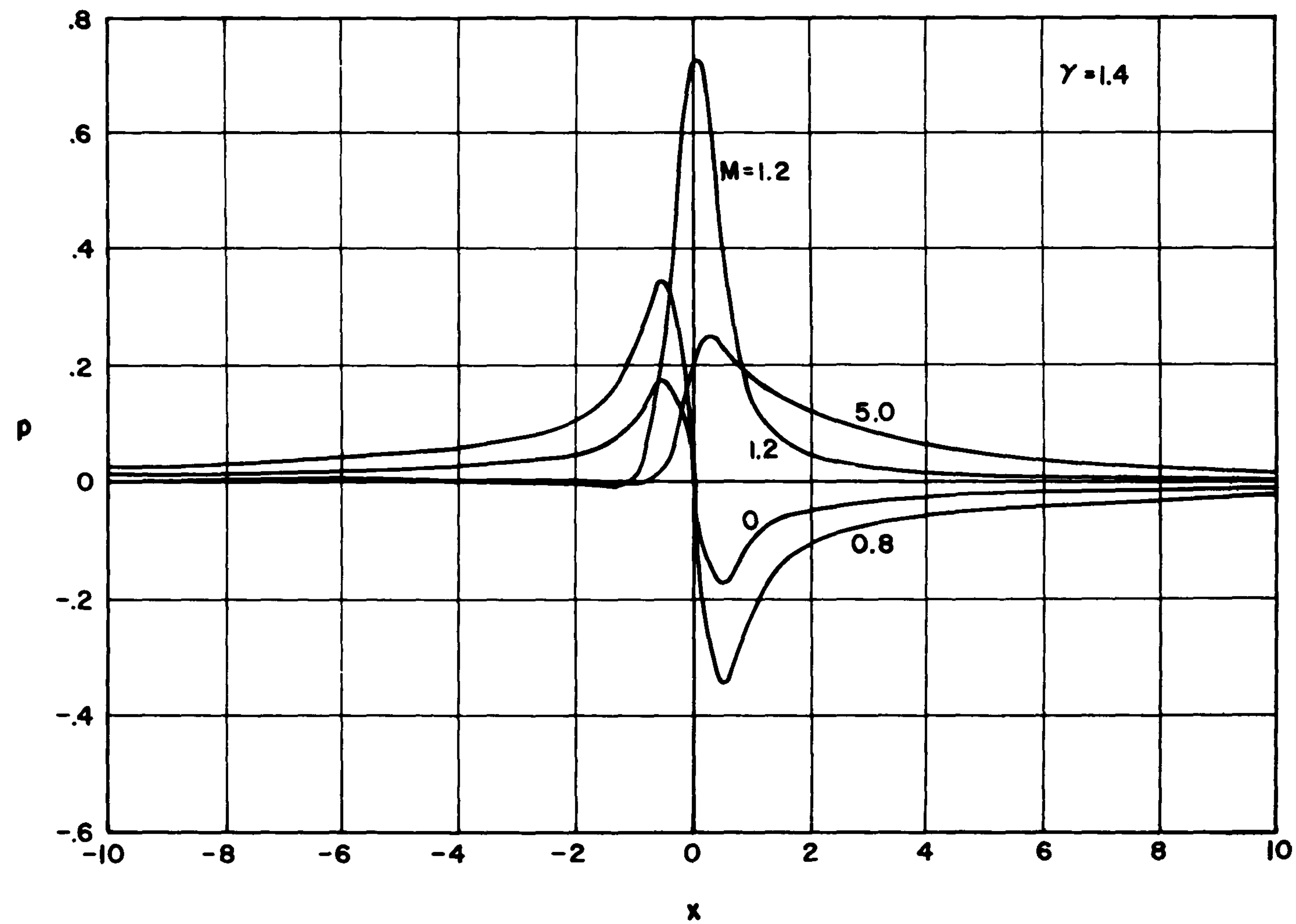

Fig. 4 This is the pressure on the wall when the flow is impeded by the magnetic field due to a linear dipole at unit distance below the wall. $\gamma = 1.4$.

7211

where, for supersonic flow, $\beta^2 = M^2 - 1$, Eqs. (18) and (26) become

$$\frac{\partial}{\partial \xi}(\beta p - v) = -\frac{\beta}{M} b_x b_y + \left[1+(\gamma-1)M^2\right]\frac{1}{M} b_y^2 \tag{53}$$

$$\frac{\partial}{\partial \eta}(\beta p + v) = \frac{\beta}{M} b_x b_y + \left[1+(\gamma-1)M^2\right]\frac{1}{M} b_y^2 \tag{54}$$

When there is just a single plate at $y = 0$ the problem has a simple solution for arbitrary field. On the wall, since $v = 0$, the pressure is just

$$\begin{aligned} p(x,0) = &-\frac{1}{M}\int_{-\infty}^{\eta-M} b_x\left\{\frac{\beta}{M}(\eta+t)\ \frac{\eta-t-M}{M}\right\} b_y\left\{\frac{\beta}{M}(\eta+t),\ \frac{\eta-t-M}{M}\right\} dt \\ &+\frac{\left[1+(\gamma-1)M^2\right]}{M\beta}\int_{-\infty}^{\eta-M} b_y^2\left\{\frac{\beta}{M}(\eta+t),\ \frac{\eta-t-M}{M}\right\} dt \end{aligned} \tag{55}$$

This expression has been evaluated for the magnetic fields discussed in the section on subsonic flow. Only the single wire result will be quoted here; it is

$$\begin{aligned} p(x,0) = &\frac{\frac{1}{2}\left[1+(\gamma-1)M^2\right]}{\beta}\left\{\frac{\frac{\pi}{2}\,\mathrm{sgn}(x+\beta)+\tan^{-1}\frac{\beta x-1}{x+\beta}}{x+\beta}\right\} \\ &-\frac{\frac{1}{2}\left[(\gamma-1)\beta x+\gamma\right]}{\beta(x^2+1)} \end{aligned} \tag{56}$$

This and the corresponding result for the linear dipole are again shown in Figs. 3 and 4. As might be expected, the symmetry present in the subsonic case is absent for supersonic flow, while the distant pressure on the wall falls off in a manner related to the fall in the magnetic field. In fact, as can be seen from Eq. (55), if the magnetic field were confined to a finite region in the neighborhood of the origin, the pressure on the wall would vanish exactly at points sufficiently far downstream so that the incoming characteristic would lie entirely outside the field region.

VI. VARIABLE CONDUCTIVITY PROBLEMS

Apart from problems involving physical boundaries, an interesting and in many ways realistic problem is that in which the conductivity is allowed to vary with position. It has been found possible to extend the solution for constant conductivity developed in this paper to the case where the conductivity is variable under the following conditions: first, for the supersonic case, with only one bounding surface at $y = 0$, the conductivity may vary in an arbitrary manner with x and y. The pressure on the wall may then be found directly as in Eq. (55) with the function $\sigma(x, y)$ included in the integral. Second, for the subsonic case, again with only one bounding surface at $y = 0$, solutions may be constructed when σ is a function of y only. This result will be demonstrated for the incompressible case, but the extension to subsonic Mach numbers is immediate.

Consider first the restricted problem where the conductivity is constant ($\sigma = 1$, say) for $0 < y < H$, and vanishes for $y > H$. The relevant equations are

$$\frac{\partial v}{\partial x} + \frac{\partial p}{\partial y} = \left.\begin{matrix} b_x b_y \\ 0 \end{matrix}\right\} \quad \left\{\begin{matrix} y < H \\ y > H \end{matrix}\right\} \tag{57}$$

$$\frac{\partial v}{\partial y} - \frac{\partial p}{\partial x} = \left.\begin{matrix} b_y^2 \\ 0 \end{matrix}\right\} \quad \left\{\begin{matrix} y > H \\ y < H \end{matrix}\right\} \tag{58}$$

Let subscripts 1 and 2 refer to the region $0 < y < H$ and $y > H$. Then, in Region 1, the particular solution given in Eqs. (29) and (30) is applicable. This may be referred to as (v_{P1}, p_{P1}). Next, define the complex function

$$F(x) = p_{P_1}(x, H) + i v_{P_1}(x, H) \tag{59}$$

It is now required to find a pair of complex functions $w_1(z) = p_1 + iv_1$ and $w_2(z) = p_2 + iv_2$ such that $w_1(z)$ is analytic in Region 1 and $w_2(z)$ is analytic

in Region 2 and such that

$$w_2(x+iH) = w_1(x+iH) + F(x) \tag{60}$$

It is convenient to require that w_1 should also be analytic for $y < 0$. Then the solution to this problem is

$$w_1(z) = \frac{-1}{2\pi i}\int_{\mathcal{C}_1} \frac{F(\zeta)d\zeta}{\zeta - z} \tag{61}$$

$$w_2(z) = \frac{1}{2\pi i}\int_{\mathcal{C}_2} \frac{F(\zeta)}{\zeta - z}\, d\zeta \tag{62}$$

where $\mathcal{C}_1$ and $\mathcal{C}_2$ are large semi-circles having the line $y = H$ for diameter and enclosing respectively the regions $y < H$ and $y > H$. At this stage the solution in Region 1 may be written symbolically as $(p_{P1}, v_{P1}) + w_1$, and in Region 2 the solution is w_2. Both p and v are now continuous across the line $y = H$, but the condition $v = 0$ on $y = 0$ has not yet been met. To accomplish this, it is only necessary to introduce a function $w(z)$, analytic in $y > 0$, and such that

$$\vartheta_w + v_{P_1} + \vartheta_{w_1} = 0 \qquad \text{on } y = 0 \tag{63}$$

and the final solution is now given by

$$\begin{aligned} &(p_{P_1}, v_{P_1}) + w_1 + w \qquad && 0 < y < H \\ &w_2 + w && y > H \end{aligned} \tag{64}$$

Two illustrative examples of this procedure are given in Levy (loc. cit.), but will not be quoted here.

Now consider the case where σ is an arbitrary function of y. Define $p_H(x, y, H)$, $v_H(x, y, H)$ to be the solution to a problem with a given

magnetic field in which $\sigma = 1$ for $y < H$ and $\sigma = 0$ for $y > H$. These quantities are just those whose derivation was explained above. Then the required solution is

$$p(x,y), v(x,y) = \sigma(\infty)\left[p_H(x,y,\infty), v_H(x,y,\infty)\right]$$

$$-\int_0^\infty \left[p_H(x,y,H), v_H(x,y,H)\right] \frac{d\sigma(H)}{dH}\, dH \tag{65}$$

To prove that this is indeed the required solution, note first that since all the solutions (p_H, v_H) are continuous for $y > 0$ and give vanishing normal velocity at the wall, the integrated solution Eq. (65) also has these properties. It remains to show that Eq. (65) satisfies the appropriate differential equations. Taking Eq. (57) for example

$$\frac{\partial v}{\partial x} + \frac{\partial p}{\partial y} = \sigma(\infty)\left[\frac{\partial v_H}{\partial x}(x,y,\infty) + \frac{\partial p_H}{\partial y}(x,y,\infty)\right]$$

$$-\int_0^\infty \left[\frac{\partial v_H}{\partial x}(x,y,H) + \frac{\partial p_H}{\partial y}(x,y,H)\right] \frac{d\sigma(H)}{dH}\, dH \tag{66}$$

But

$$\frac{\partial v_H}{\partial x} + \frac{\partial p_H}{\partial y} = \begin{array}{ll} b_x b_y & (H > y) \\ 0 & (H < y) \end{array} \tag{67}$$

Thus

$$\frac{\partial v}{\partial x} + \frac{\partial p}{\partial x} = \sigma(\infty) b_x b_y - \int_0^\infty b_x b_y \frac{d\sigma(H)}{dH}\, dH = \sigma(y)\, b_x b_y \tag{68}$$

as required; similarly for Eq. (58).

VII. CONCLUSIONS

A variety of small perturbation magnetohydrodynamic flow problems have been solved by methods which present the results in an easily assimilable form. A few have been presented in detail here, others referred to. All these problems could, in principle, have been solved by the more widely applicable transform methods, but the results obtained by such methods are rarely suitable for more than numerical computation of a few cases. In the simple cases quoted here, the effects of compressibility and the effects of different values of γ are readily visible together with the effects of variable conductivity (as specified in the last section).

The general nature of the results is not, on the whole, surprising, but tends to agree with other work dealing with similar problems by more involved techniques. However, the relative importance of the electromagnetic and the induced pressure forces is brought out clearly. More specifically, there is a force on the wall due to the induced pressure field which, like the electromagnetic force on the coil, is of the first order in the interaction parameter. In the problems considered in this paper, the nature of the boundary has the obvious result that this pressure force can only contribute a lift, but it is certainly true that for bodies not consisting only of planes parallel to the flow, the induced pressure would also contribute to the drag. In the special configurations considered here, for subsonic flows, the pressure distribution is symmetrical about the wire or dipole; clearly this will only happen for highly symmetrical configurations and has no general validity. In addition to the net forces exerted by the pressure distribution, a resultant moment will also be found in most cases. Reference to Fig. 4 shows the interesting result that, for the case considered, the moment would appear to be "nose down" for subsonic flow, and "nose up" for supersonic flow.

Finally, it should be emphasized that the weakest point in this, as in all comparable analyses, is the assumption (for compressible flows) of uniform scalar conductivity. It is hoped that in future problems of this nature it will prove possible to treat the conductivity as a realistic function of the temperature in order to obtain results more closely representative of real flows.

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Avco-Everett Research Laboratory, Everett, Massachusetts
EXACT SOLUTIONS TO A CLASS OF LINEARIZED MAGNETOHYDRODYNAMIC FLOW PROBLEMS, by R. H. Levy. November 1961. 25 p. incl. illus. (Avco-Everett Research Report 124; BSD-TDR-62-42)
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Unclassified report

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